Recitation 10

October 29, 2015

Review

Algorithm for finding a matrix of linear transformation in two bases. Namely, suppose you have a transformation $T\colon V \to W$ from n-dimensional space to m-dimensional one. Let $\mathcal{B} = \{v_1, \dots, v_n\}$ be a basis in V and $\mathcal{C} = \{w_1, \dots, w_m\}$ be a basis in W. You need to find the matrix $M = [T]_{\mathcal{B},\mathcal{C}}$.

1. Find $T(v_1), \dots, T(v_n)$.
2. Find the coordinates $[T(v_1)]_{\mathcal{C}}, \dots, [T(v_n)]_{\mathcal{C}}$ of $T(v_1), \dots, T(v_n)$ in the basis $\mathcal{C}$.
3. Put that into matrix: $M = [[T(v_1)]_{\mathcal{C}}, \dots, [T(v_n)]_{\mathcal{C}}]$.

Algorithm to find matrix of $A\colon \mathbb{R}^n \to \mathbb{R}^m$ in some bases. The same thing as above really. If $\mathcal{B} = \{v_1, \dots, v_n\}$ is a basis of $\mathbb{R}^n$, put these vectors into a matrix $P = [v_1 \dots v_n]$. If $\mathcal{C} = \{w_1, \dots, w_m\}$ is a basis of $\mathbb{R}^m$, put these vectors into a matrix $Q = [w_1 \dots w_m]$. Then the matrix of A in non-standard bases $\mathcal{B}, \mathcal{C}$ is $M = Q^{-1}AP$.
If $n = m$ and $\mathcal{B} = \mathcal{C}$: the same stuff really. It's just now $P = Q$, so $M = P^{-1}AP$.
Notice: diagonalization is a particular case of that. If $P = [v_1 \dots v_n]$ is a basis of **eigenvectors** of A, then $D = P^{-1}AP$ is the diagonalization of A corresponding to $\{v_1, \dots, v_n\}$.

Problems

Problem 1. Let $A = \begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$. Find its eigenvalues and eigenvectors.

Find a matrix P, and a matrix C of the form $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ such that $A = PCP^{-1}$.

Problem 2. Let $\mathcal{B} = \{v_1, v_2, v_3\}$ be three linearly independent vectors in $\mathbb{R}^{13}$, and let $H = Span(v_1, v_2, v_3)$. Let $\mathcal{C} = \{v_1, v_2 - v_3, v_1 + v_2 + v_3\}$.

- Prove that $\mathcal{C}$ is another basis for H.
- Let $T\colon H \to H$ be the linear transformation such that $T(v_1) = v_3 - v_2$, $T(v_2) = v_1 - 2v_2 + v_3$ and $T(v_3) = 3v_1 - v_2 - 2v_3$. Find the matrix of linear transformation T in relative to the basis $\mathcal{B}$.
- Find the matrix of T relative to the basis $\mathcal{C}$.
- Find rank of T.

Problem 3. Let $\mathcal{B} = \{v_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}\}$ and $\mathcal{C} = \{w_1 = \begin{bmatrix} -2 \\ 5 \end{bmatrix}, w_2 = \begin{bmatrix} 3 \\ -8 \end{bmatrix}\}$

- Prove that both $\mathcal{B}, \mathcal{C}$ are bases of $\mathbb{R}^2$.
- Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation. You know that relative to the basis $\mathcal{B}$ the matrix of T is $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$. Find the matrix of T relative to the basis $\mathcal{C}$.

Problem 4. Let $v = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. Describe the set of vectors $W = \{w \in \mathbb{R}^3 \mid w \cdot v = 0\}$ orthogonal to v.
Is W a subspace of $\mathbb{R}^3$?
Find a basis of W.

Problem 5. Show that if $x \in W$ and $x \in W^\perp$ then $x = 0$.

Problem 6. Let $y = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $u = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$. Write y as a sum of two orthogonal vectors, one in $Span(u)$ and one in $Span(u)^\perp$.

Problem 7. Let $u = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Is the set $\{u, v\}$ orthogonal? Is it orthonormal? If not, then normalize to make it so.

Problem 8. Let U be a $n \times n$ matrix with orthonormal columns. Why is U necessarily invertible?

Problem 9. Let U, V be two $n \times n$ orthogonal matrices. Why is UV also orthogonal?

Problem 10. Let $u \in \mathbb{R}^n$ be a non-zero vector, and denote $L = Span(u)$. Prove that the map $T \colon \mathbb{R}^n \to L$ given by $T(y) = proj_L(y)$ is a linear transformation.